SE 422 Advanced Photogrammetry

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Mathematical Concepts of Photogrammetry

Contents:

- Coordinate Reference Frames
- 3D Rotation Matrix (M)
- Collinearity Equation
- Coplanarity Equation
- 3D Transformation (7-Parameter Transformation)
-

Coordinate Reference Frames:

In Photogrammetry, we have two primary reference coordinates system:

- Image Space Coordinate system $(x - x_0, y - y_0, -f)$
- Object Space Coordinate system, which is the 3D region covered by the photograph or image

These coordinates reference systems will always be considered to be Cartesian and right-handed

Coordinate Reference Frames:

Image Space Coordinate System:

The origin of this frame is the principal point (x_0, y_0)

By placing the 3-D origin at the perspective center, this yields to referred as the *sensor coordinates* or *image space coordinates*

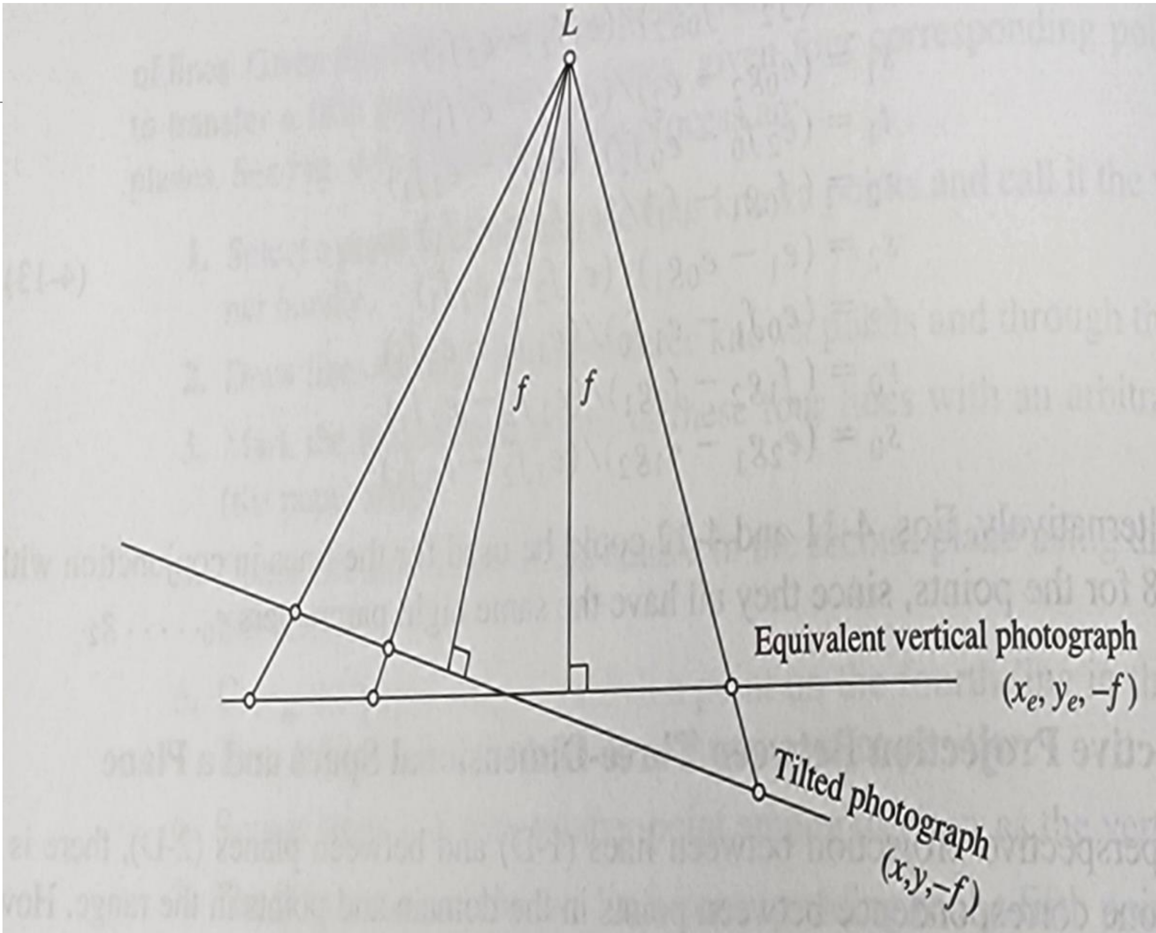


Figure 4-4 Original tilted photograph and equivalent vertical photograph.

This figure is taken from “E.Mikhail, J.Bethel, and C.McGlone “Introduction to Modern Photogrammetry””

Sensor Model (Interior Orientation Parameter IOP)

The Interior Orientation defines the camera(sensor) characteristics required for reconstructing the object space bundle of rays from the corresponding image points.

In a frame camera, these characteristics would include at least:

- the focal length (or principal distance)
- location of the principal point
- A description of the lens distortion

Sensor Model (Interior Orientation Parameter IOP)

The principal point is usually given with respect to the coordinate axes defined by fiducial marks.

For digital cameras, the principal point is usually given with respect to the image row-column coordinate system

Interior Orientation parameters are: ***focal length (f), principal point (*** x_0, y_0 ***), and lens distortions***

Platform Model (Exterior Orientation EO)

This model establishes the position and orientation of the bundle of rays with respect to the object space coordinate system

Each bundle requires 6 independent elements (3 for position and 3 for orientation)

For frame cameras: one bundle represents the entire image

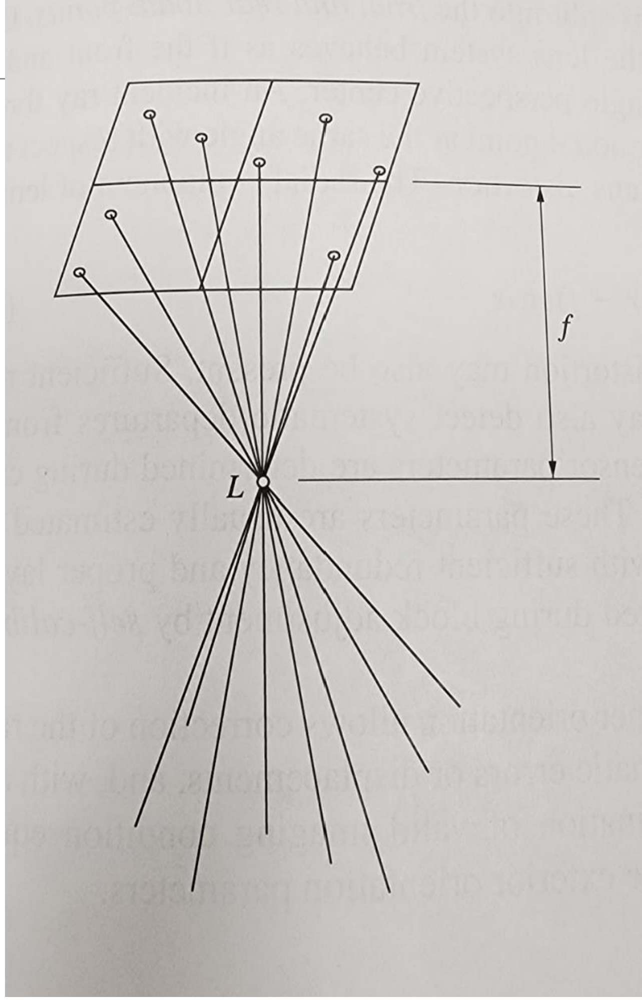
In case of a linear sensor: each line defines a new bundle (with its own 6 elements of EO)

Platform Model (Exterior Orientation EO)

For each bundle of rays: the three elements of position fix the location of the perspective center (point (L) in the figure)

These three elements usually referred to as the exposure station(camera station), and they are expressed as:

$$L = \begin{bmatrix} X_L \\ Y_L \\ Z_L \end{bmatrix}$$



This figure is taken from "E.Mikhail, J.Bethel, and C.McGlone "Introduction to Modern Photogrammetry""

Platform Model (Exterior Orientation EO)

With this point (L), the rays can still take any orientation in space.

To describe this orientation in the object space, 3 independent parameters will be sufficient (using the geometry)

These 3 parameters define the 3D rotation matrix (M), which relates the object space and image space systems.

These 3 parameters are ω, ϕ, and κ, representing the rotation around the X-axis, Y-axis, and Z-axis, respectively.

Then, the EOPs for any image are: (X_L, Y_L, and Z_L) and (ω, ϕ, and κ)

3D Rotation Matrix (M)

3D Rotation Matrix (M)

In 3D, there are 3 elementary rotations

One about each axis (X, Y, and Z axis)

To construct the rotation matrix, you need three angles, which are $(\omega, \phi, and\ \kappa)$

3D Rotation Matrix (M)

First, perform the rotation around the X-axis ($M\omega$)

$$M_\omega = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix}$$

Positive rotation will advance the +Y axis towards +Z axis.

3D Rotation Matrix (M)

Then, make the rotation around the Y-axis (M_ϕ)

$$M_\phi = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix}$$

Positive rotation will advance the +Z axis towards +X axis.

3D Rotation Matrix (M)

Lastly, make the rotation around the Z-axis (M_κ)

$$M_\kappa = \begin{bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Positive rotation will advance the +X axis towards +Y axis.

3D Rotation Matrix (M)

Then, the rotation matrix (M) can be constructed as a set of sequential rotations:

$$M = M_{\kappa} M_{\phi} M_{\omega}$$

Since $M_{\kappa}, M_{\phi}, and\ M_{\omega}$ are each orthogonal, M is also orthogonal matrix (what does that mean?)

Elements of the 3D Rotation Matrix (M)

$$M = \begin{bmatrix} \cos\phi\cos\kappa & \cos\omega\sin\kappa + \sin\omega\sin\phi\cos\kappa & \sin\omega\sin\kappa - \cos\omega\sin\phi\cos\kappa \\ -\cos\phi\sin\kappa & \cos\omega\cos\kappa - \sin\omega\sin\phi\sin\kappa & \sin\omega\cos\kappa + \cos\omega\sin\phi\sin\kappa \\ \sin\phi & -\sin\omega\cos\phi & \cos\omega\cos\phi \end{bmatrix}$$

Suppose that the elements of the rotation matrix were given, then the rotation angles can be computed as:

$$m_{31} = \sin\phi \Rightarrow \phi = \sin^{-1}(m_{31})$$

$$\frac{m_{32}}{m_{33}} = \frac{-\sin\omega\cos\phi}{(\cos\omega\cos\phi)} = -\tan\omega \Rightarrow\Rightarrow\Rightarrow \omega = \tan^{-1}(\frac{-m_{32}}{m_{33}})$$

$$\frac{m_{21}}{m_{11}} = \frac{-\cos\phi\sin\kappa}{\cos\phi\cos\kappa} = -\tan\kappa \Rightarrow \Rightarrow \Rightarrow \kappa = \tan^{-1}\left(\frac{-m_{21}}{m_{11}}\right)$$

Don’t forget to return the angles (ω and κ) to their appropriate quadrants

Next Lecture:

- We will continue this chapter about the Mathematical concepts of Photogrammetry, covering:
  - Collinearity
  - Coplanarity
  - 7-Parameter Transformation and derive the partial derivatives for this transformation
- Quiz-4 will be on Sunday, 22/1/2023, at the beginning of the lecture (DON'T BE LATE).

Collinearity Equation

Collinearity Equations

The fundamental characteristics of frame imaging is that:

- the perspective center
- image point
- and the corresponding object point

all lie on a line in space.

This line can be defined as vector components in the image coordinate system or vector components in the object space coordinate system

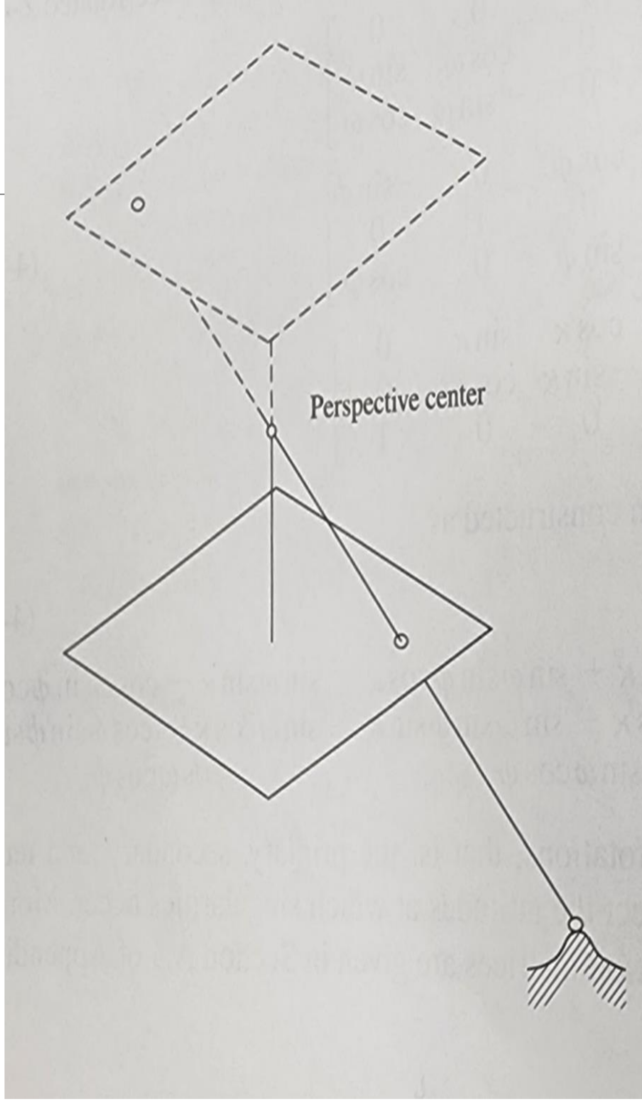


Figure 4-8 Imaging geometry for a single point on a frame photograph.

This figure is taken from "E.Mikhail, J.Bethel, and C.McGlone "Introduction to Modern Photogrammetry""

Collinearity Equations

These image coordinates and object coordinates will be related using the EOP as:

$$\begin{bmatrix} x \\ y \\ -f \end{bmatrix} = M \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

From this equation and what we have covers in the 2D transformations, what are the missing parameters?

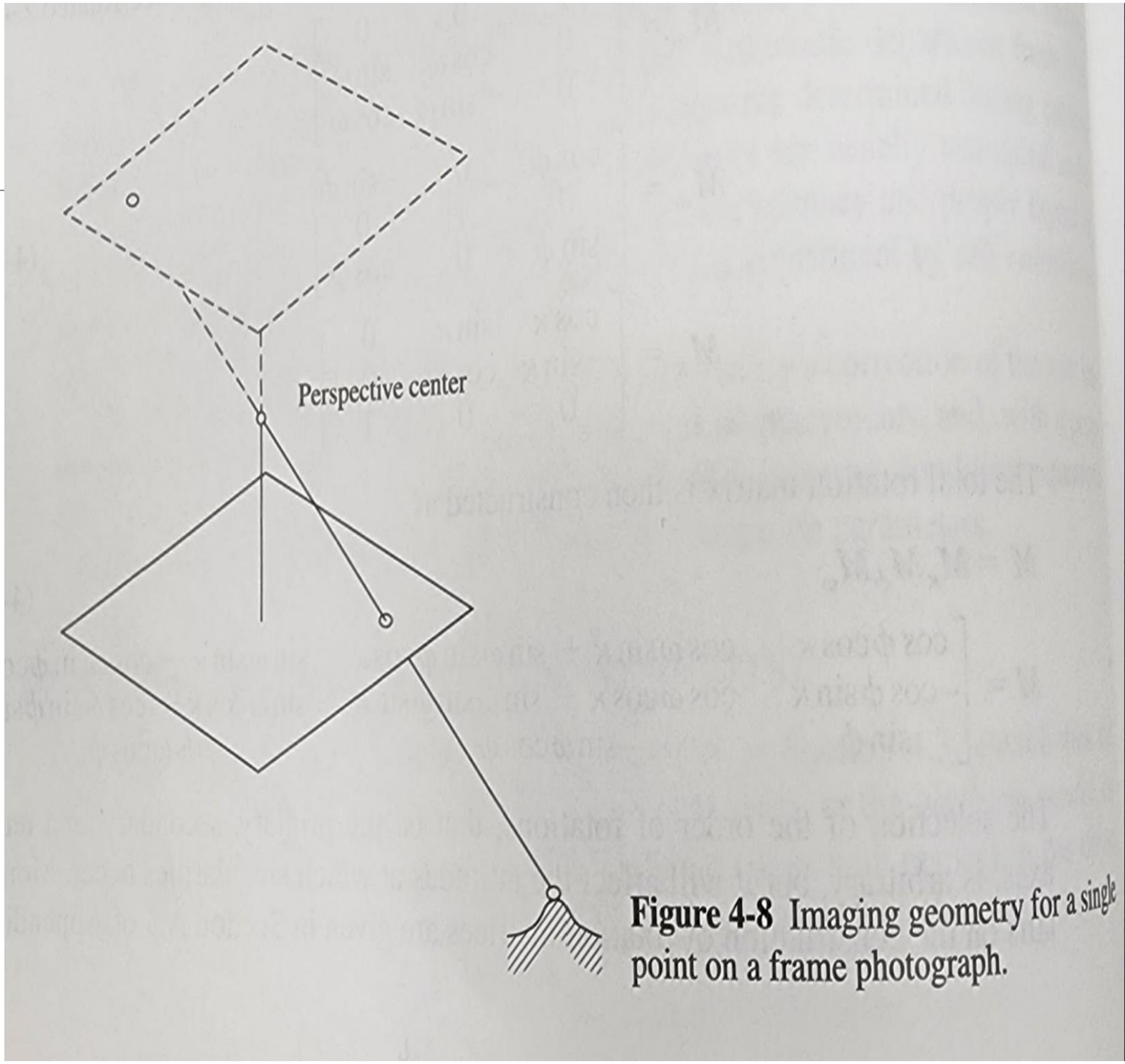


Figure 4-8 Imaging geometry for a single point on a frame photograph.

This figure is taken from "E.Mikhail, J.Bethel, and C.McGlone "Introduction to Modern Photogrammetry""

Collinearity Equations

the previous equation assumes that the origin of the two coordinates system coincides (matches).

However, in fact, they don't match. So, we need to introduce shift terms to place a local origin of object space at the perspective center (L).

We also need to introduce a scale parameter (λ) to express the difference in magnitude between the image space vector and the corresponding object space vector.

$$\begin{bmatrix} x'' \\ y'' \\ -f \end{bmatrix} = \lambda M \begin{bmatrix} X - X_L \\ Y - Y_L \\ Z - Z_L \end{bmatrix}$$

Note that you have to use the corrected (refined) image coordinates.

$x' = x - x_0$ and $y' = y - y_0$

$x'' = x' + \Delta dx$ and $y'' = y' + \Delta dy$

Where $\Delta dx \ and \ \Delta dy$ are the total distortion in x and y coordinates, respectively.

Collinearity Equations

To delete the nuisance(scale) factor (λ):

$$\begin{bmatrix} x'' \\ y'' \\ -f \end{bmatrix} = \lambda \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} X - X_L \\ Y - Y_L \\ Z - Z_L \end{bmatrix}$$

After multiplying the matrix and vectors on the RHS:

$$\begin{cases} x'' = \lambda \left(m_{11}(X - X_L) + m_{12}(Y - Y_L) + m_{13}(Z - Z_L) \right) \\ y'' = \lambda \left(m_{21}(X - X_L) + m_{22}(Y - Y_L) + m_{23}(Z - Z_L) \right) \\ -f = \lambda \left(m_{31}(X - X_L) + m_{32}(Y - Y_L) + m_{33}(Z - Z_L) \right) \end{cases}$$

Collinearity Equations

To delete the nuisance(scale) factor (λ):

Divide the 1st and 2nd equation by the 3rd equation:

$$x'' = -f\,\frac{\left(m_{11}(X - X_L) + m_{12}(Y - Y_L) + m_{13}(Z - Z_L)\right)}{\left(m_{31}(X - X_L) + m_{32}(Y - Y_L) + m_{33}(Z - Z_L)\right)}$$

$$y'' = -f\,\frac{\left(m_{21}(X - X_L) + m_{22}(Y - Y_L) + m_{23}(Z - Z_L)\right)}{\left(m_{31}(X - X_L) + m_{32}(Y - Y_L) + m_{33}(Z - Z_L)\right)}$$

Collinearity Equations

Since the rotation matrix is orthogonal $M^{-1} = M^T$

The collinearity equation can also be rewritten as:

$$\begin{bmatrix} X - X_L \\ Y - Y_L \\ Z - Z_L \end{bmatrix} = \frac{1}{\lambda} M^T \begin{bmatrix} x'' \\ y'' \\ -f \end{bmatrix}$$

we can repeat the same previous steps to delete the scale factor

$$X - X_L = Z - Z_L \cdot \left(\frac{\left(m_{11}x'' + m_{21}y'' + m_{31}(-f)\right)}{\left(m_{13}x'' + m_{23}y'' + m_{33}(-f)\right)} \right)$$

$$Y - Y_L = Z - Z_L \cdot \left(\frac{\left(m_{12}x'' + m_{22}y'' + m_{32}(-f)\right)}{\left(m_{13}x'' + m_{23}y'' + m_{33}(-f)\right)} \right)$$

Collinearity Equations

As we have covered in SE331:

Least-squares technique will be used to estimate the parameters of the collinearity equation

(Can you point out these parameters? Can you also tell if the collinearity equation is Linear or Nonlinear?)

Collinearity Equations

Since it is nonlinear, this will force us to use Taylor's Series Approximations

Then, collinearity equation can be rewritten as:

$$let \Rightarrow \begin{bmatrix} U \\ V \\ W \end{bmatrix} = M \begin{bmatrix} X - X_L \\ Y - Y_L \\ Z - Z_L \end{bmatrix}$$

$$then \Rightarrow condition\ equations\ can\ be\ written\ as:$$

$$F_1 = x'' + f\left(\frac{U}{W}\right)$$

$$F_1 = y'' + f\left(\frac{V}{W}\right)$$

Collinearity Equations

Then, the condition equations can be linearized as:

$$Av + B\Delta = f \ or \ K$$

(This form of Equation is defined in Observation Adjustment as GLS)

Note that f in the RHS of the previous equation is not the focal length. It is the misclosure vector, and can be computed as:

$$f \ or \ K = \begin{bmatrix} -F_1(l^0, x^0) \\ -F_1(l^0, x^0) \end{bmatrix}$$

Collinearity Equations

A is the partial derivatives matrix of condition equations wrt observations

B is the partial derivatives matrix of condition equations wrt parameters (unknowns)

Δ is the parameters (unknowns) vector or some time called (space vector)

v is the observations residuals

Coplanarity Equation

Coplanarity Equations

As illustrated in the figure:

Suppose we have two photos that are relatively oriented with respect to each other

Then, the conjugate object space rays defined by the image points and their perspective centers will intersect exactly.

This intersection defines the object's Space Position.

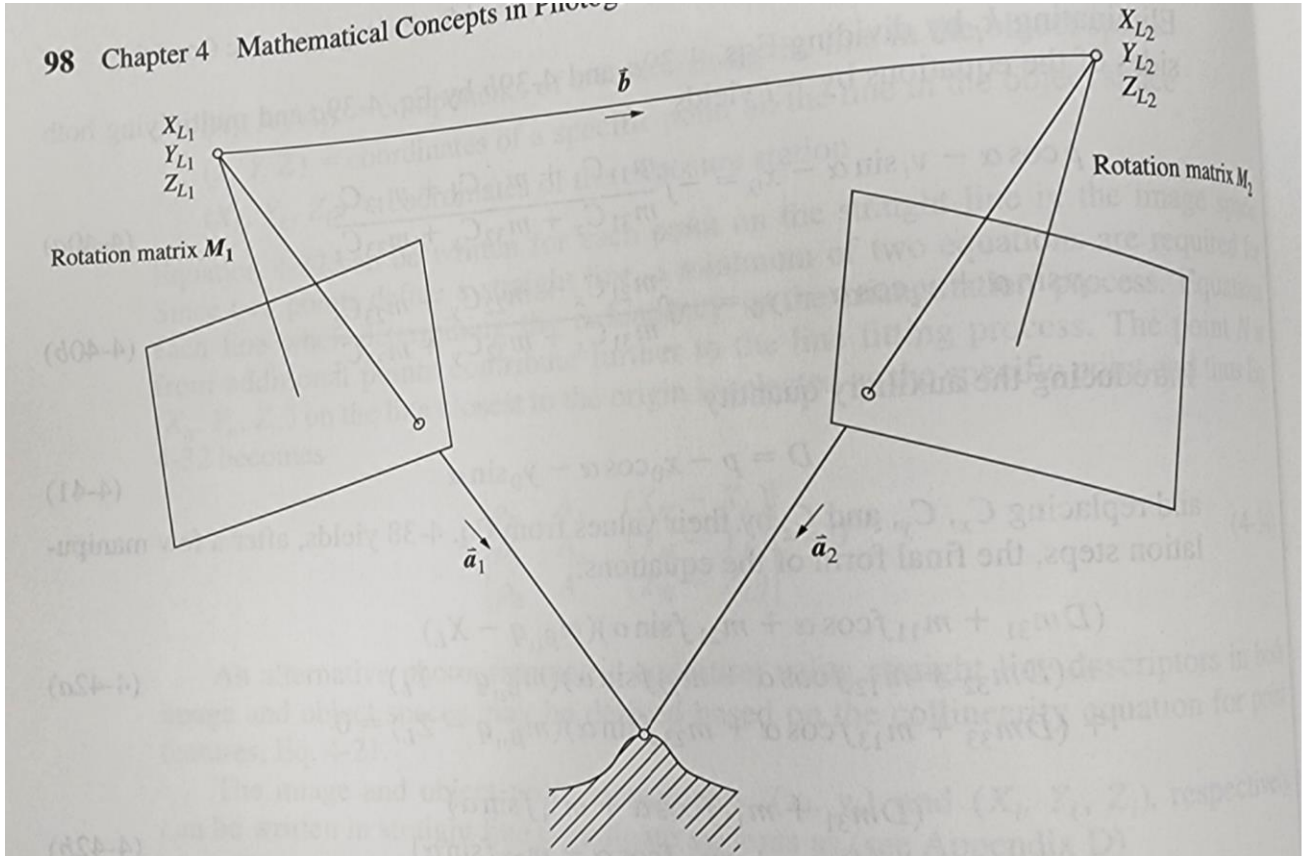


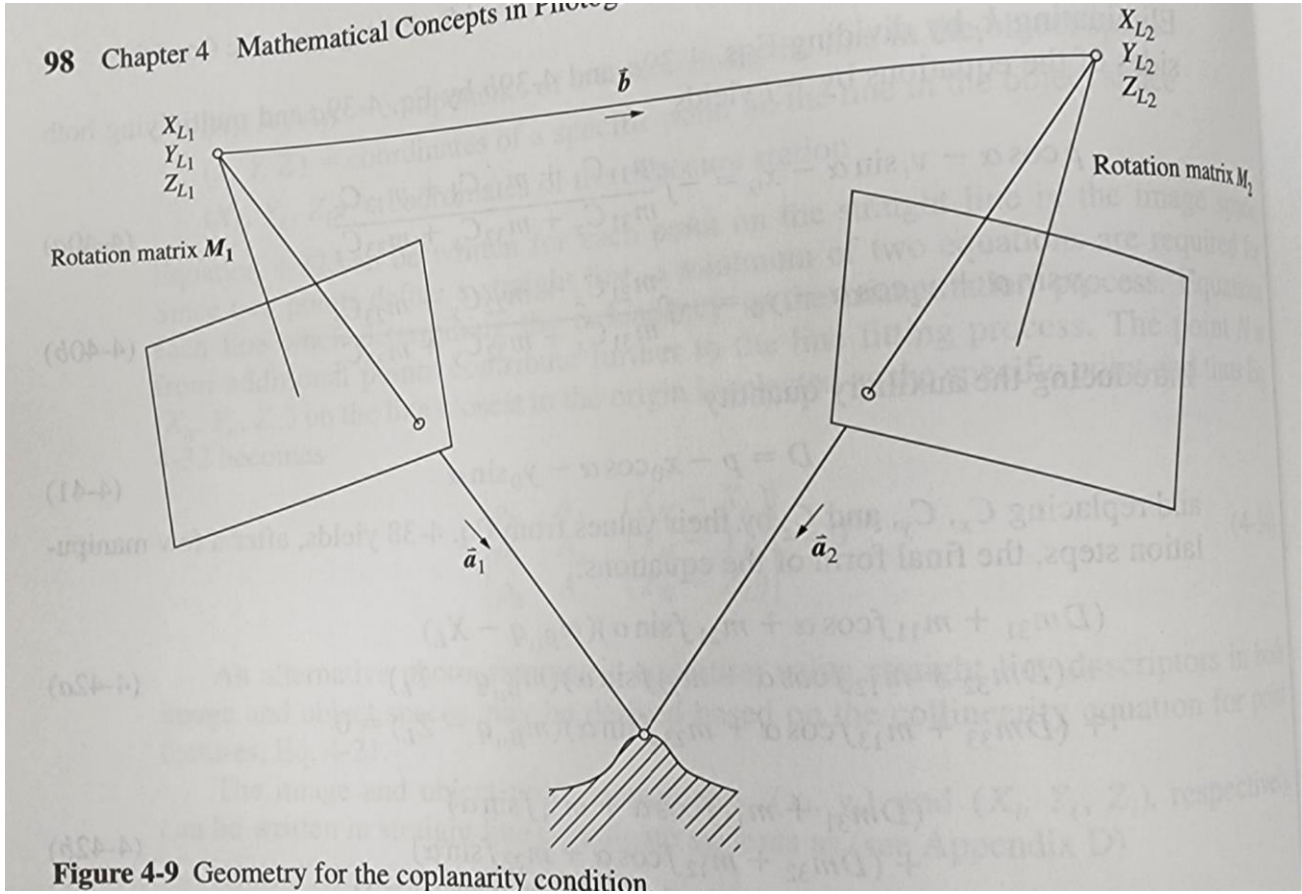
Figure 4-9 Geometry for the coplanarity condition

This figure is taken from "E.Mikhail, J.Bethel, and C.McGlone "Introduction to Modern Photogrammetry""

Coplanarity Equations

These two points rays and the base-line between the two perspective centers define a plane.

The equation that enforces this relationship between the three sides of this plane is the Coplanarity Equation



This figure is taken from “E.Mikhail, J.Bethel, and C.McGlone “Introduction to Modern Photogrammetry””

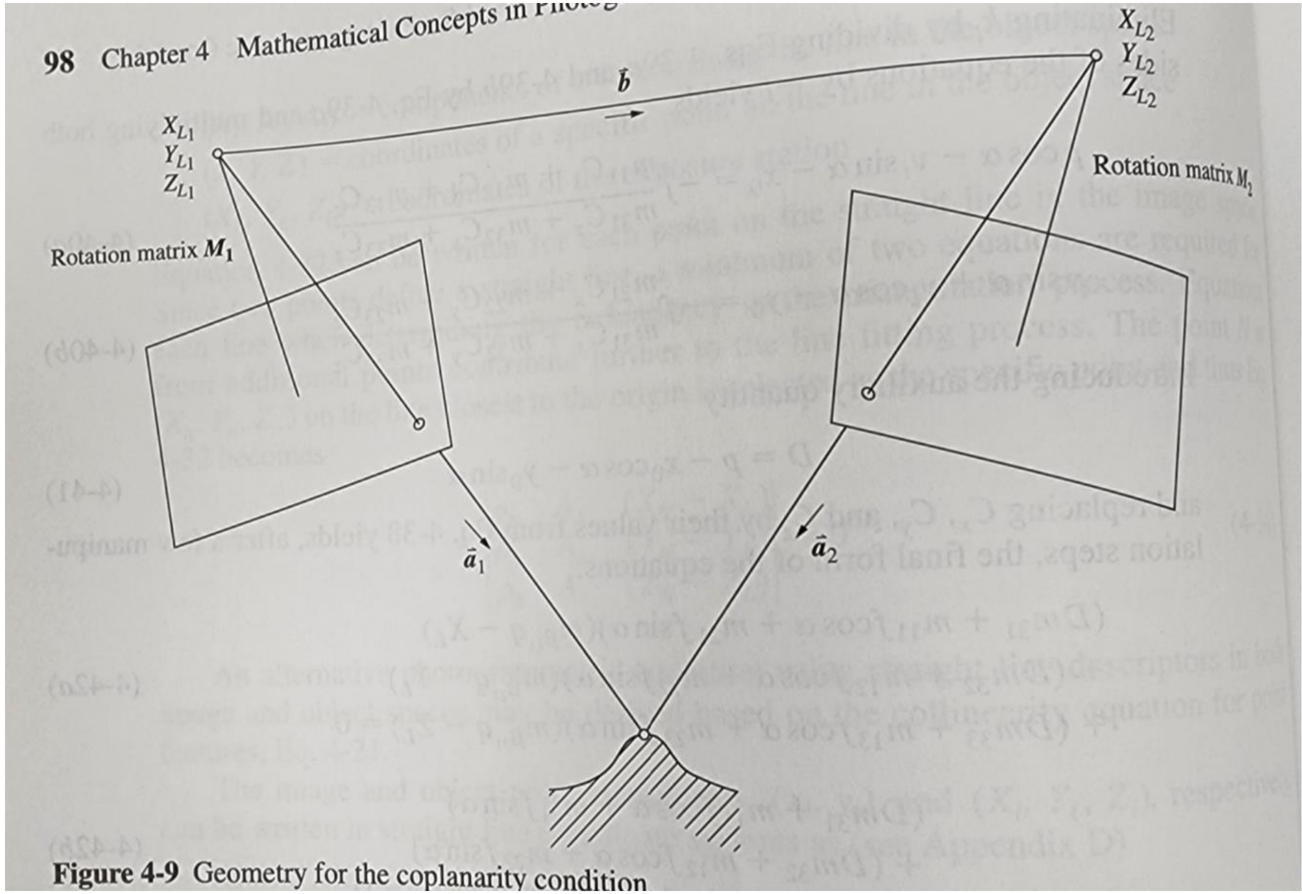
Coplanarity Equations

The Coplanarity equation for the figure on the right can be written as:

$$\vec{b} \cdot (\vec{a}_1 \times \vec{a}_2) = 0$$

Where, vector b is:

$$\vec{b} = \begin{bmatrix} b_X \\ b_Y \\ b_Z \end{bmatrix} = \begin{bmatrix} X_{L_2} - X_{L_1} \\ Y_{L_2} - Y_{L_1} \\ Z_{L_2} - Z_{L_1} \end{bmatrix}$$



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Coplanarity Equations

The Coplanarity equation for the figure on the right can be written as:

$$\vec{b} \cdot (\vec{a}_1 \times \vec{a}_2) = 0$$

Where, vector $\vec{a}_1$ is:

$$\vec{a}_1 = \begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = M_1^T \begin{bmatrix} x_1 - x_{0_1} \\ y_1 - y_{0_1} \\ -f \end{bmatrix}$$

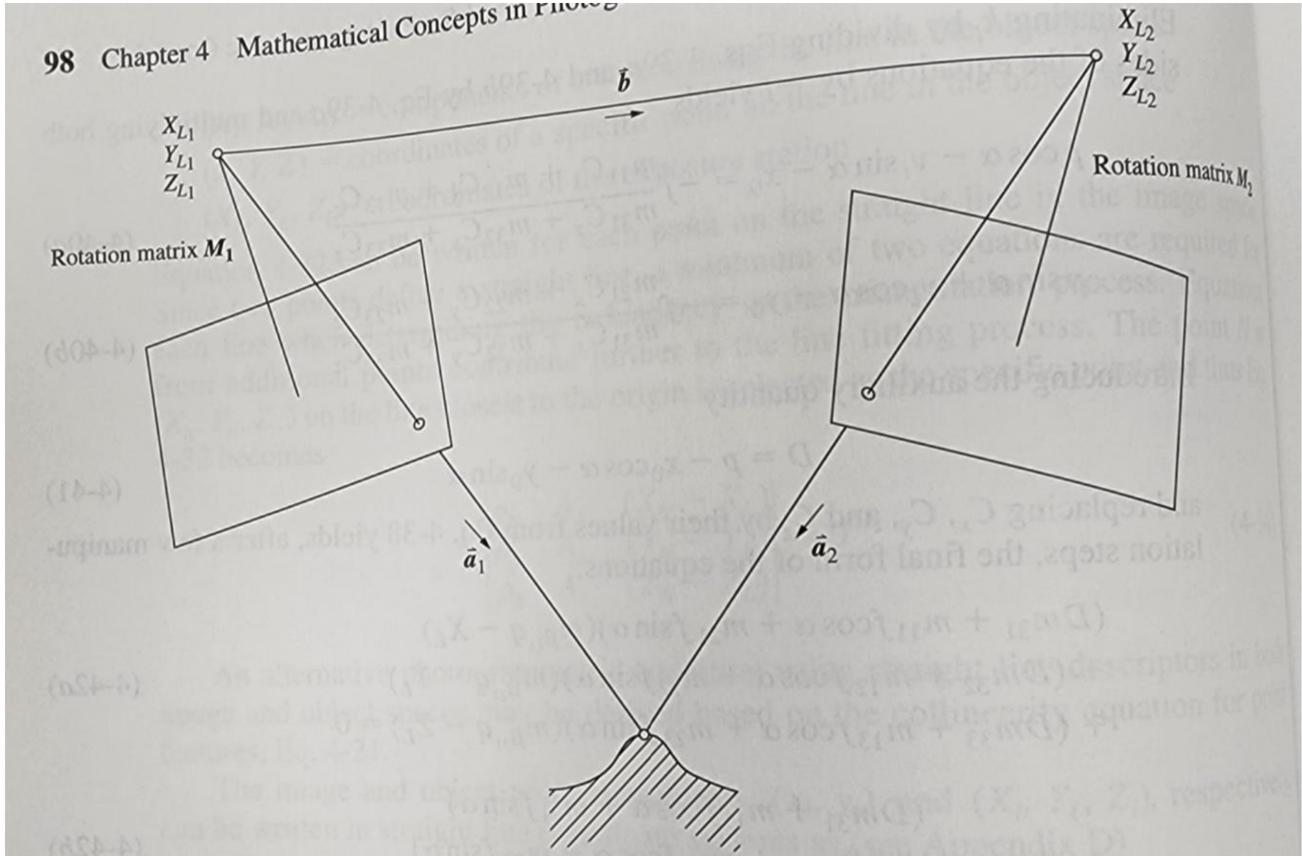

Figure 4-9 Geometry for the coplanarity condition

This figure is taken from "E.Mikhail, J.Bethel, and C.McGlone "Introduction to Modern Photogrammetry""

Coplanarity Equations

The Coplanarity equation for the figure on the right can be written as:

$$\vec{b} \cdot (\vec{a}_1 \times \vec{a}_2) = 0$$

Where, vector $\vec{a}_2$ is:

$$\vec{a}_2 = \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = M_2^T \begin{bmatrix} x_2 - x_{0_2} \\ y_2 - y_{0_2} \\ -f \end{bmatrix}$$

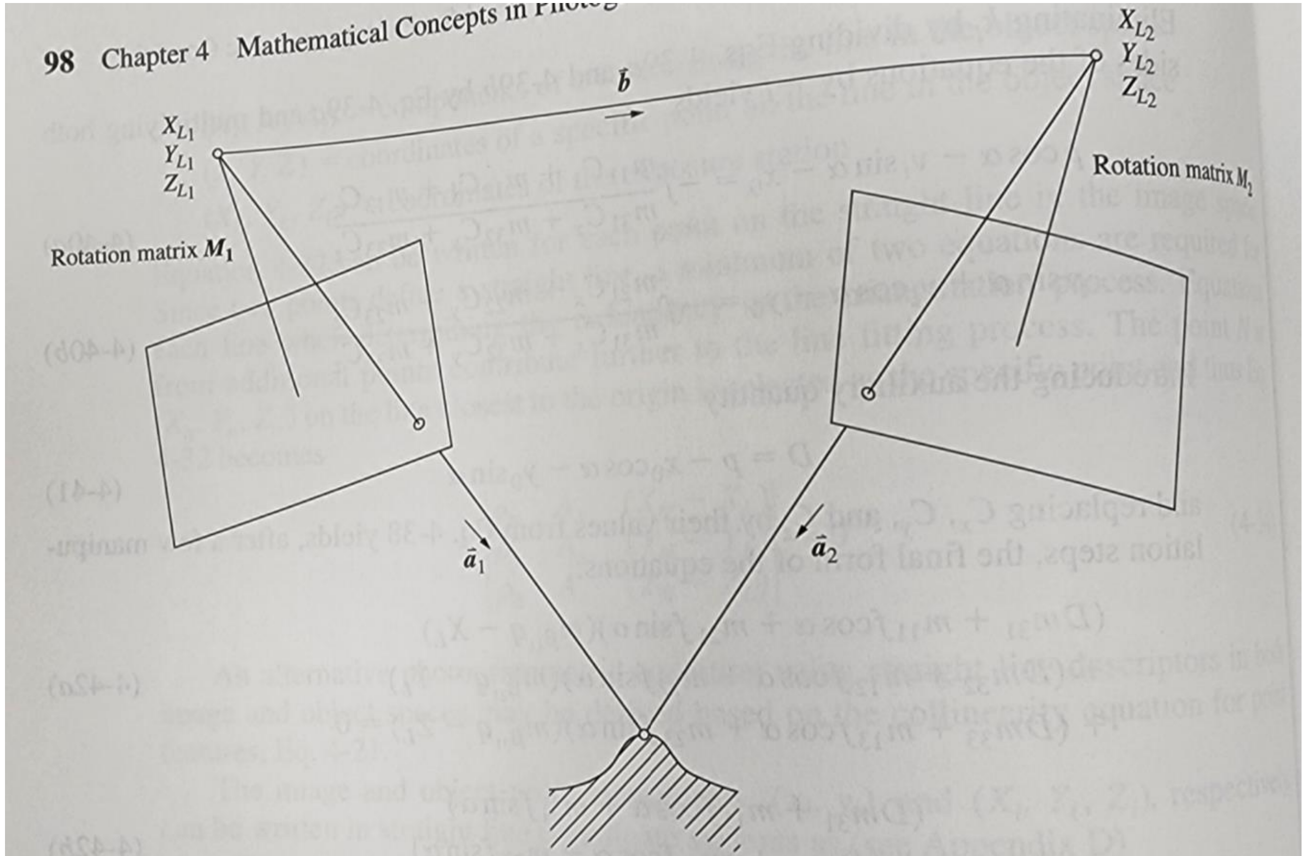

Figure 4-9 Geometry for the coplanarity condition

This figure is taken from "E.Mikhail, J.Bethel, and C.McGlone "Introduction to Modern Photogrammetry""

Coplanarity Equations

The coplanarity equation can also be written in the determinant form using the vector components:

$$F = \begin{vmatrix} b_X & b_Y & b_Z \\ u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{vmatrix} = 0$$

The coplanarity equation is mainly used to do relative orientation between pairs of photos

This can be done by fixing 7 parameters among the 12 parameters (EOPs of two photos), and solving for the remaining 5 parameters

Coplanarity Equations

The advantage of using that model is no need to have approximated space coordinates generated in the model since the equations have no object coordinates.

However, using this model in a triplet using only coplanarity condition does not guarantee that the three rays will intersect in a single point

To enforce the three rays to intersect at the same point, the Scale Restraint Equation is used.

Next week:

- We will derive the partial derivatives for the collinearity and coplanarity equations
- If the MATLAB is still not working, we might have a second midterm exam by the end of this month
- We will talk about the most useful functions of Photogrammetry, which are Space Intersection, Space Resection, and Bundle Block Adjustment.
- There will be a quiz next week (from the end of quiz 3 topics until today’s lecture)

3D Conformal Transformation

3D Conformal Transformation

- This type of transformation is also called the 7-parameter Transformation
- Applications of 3D Conformal Transformation:
  - Transform the arbitrary stereomodel coordinates to object space systems
  - Determining the initial approximations (whenever the situation is not straightforward)

- As we recall from the 2D transformation, conformal transformation means the shape is preserved.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda M \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

Partial Derivatives of the 7-Parameter Transformation

Recall from the previous slides:

$$Av + B\Delta = f$$

The most common case is when the image coordinates are considered the observation, the IOPs are known, and the remaining variables are considered unknown

Remaining variables are EOPs and the 3D information of the object point

Then, the least-squares can be written as:

$$v + B\Delta = f$$

Partial Derivatives of the 7-Parameter Transformation

$$v + B\Delta = f$$

The size of each matrix or vector:

$v: 3nx1$ $B: 3nxu$ $\Delta: ux1$ $f: 3nx1$ ➔ $f = -F$

As we have mentioned previously, there are many ways to find the partial derivatives of a function

One way is by making the analytical partial derivatives

Another way is by finding them numerically

Partial Derivatives of the 7-Parameter Transformation

Assume we have a 7-parameter model, where the parameters are the scale, EOPs, and the object point coordinates

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda M \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

Then, the condition equation for that model can be written as:

$$F = \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \lambda M \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} - \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

Partial Derivatives of the 7-Parameter Transformation

Here I will derive one angle and one shift parameters.

The rest will be your HW#4.

For tx:

$$\frac{\partial F}{\partial t_x} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

For ω:

$$\frac{\partial F}{\partial \omega} = -\lambda \frac{\partial M}{\partial \omega} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Where, $\frac{\partial M}{\partial \omega} = M_\kappa M_\phi \frac{\partial M_\omega}{\partial \omega} \rightarrow \frac{\partial M_\omega}{\partial \omega} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\omega & \cos\omega \\ 0 & -\cos\omega & -\sin\omega \end{bmatrix}$

Partial Derivatives of the 7-Parameter Transformation

Where, $\frac{\partial M}{\partial \omega} = M_\kappa M_\phi \frac{\partial M_\omega}{\partial \omega}$ ➔ $\frac{\partial M_\omega}{\partial \omega} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\omega & \cos\omega \\ 0 & -\cos\omega & -\sin\omega \end{bmatrix}$

Then, $\frac{\partial M}{\partial \omega}$

$$= \begin{bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\omega & \cos\omega \\ 0 & -\cos\omega & -\sin\omega \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \cos\kappa\cos\omega\sin\phi - \sin\kappa\sin\omega & \cos\omega\sin\kappa + \cos\kappa\sin\phi\sin\omega \\ 0 & -\cos\kappa\sin\omega - \cos\omega\sin\kappa\sin\phi & \cos\kappa\cos\omega - \sin\kappa\sin\phi\sin\omega \\ 0 & -\cos\phi\cos\omega & -\cos\phi\sin\omega \end{bmatrix}$$

Partial Derivatives of the 7-Parameter Transformation

$$\frac{\partial F}{\partial \omega} = -\lambda \begin{bmatrix} 0 & \cos\kappa\cos\omega\sin\phi - \sin\kappa\sin\omega & \cos\omega\sin\kappa - \cos\kappa\sin\phi\sin\omega \\ 0 & -\cos\kappa\sin\omega - \cos\omega\sin\kappa\sin\phi & \cos\kappa\cos\omega + \sin\kappa\sin\phi\sin\omega \\ 0 & -\cos\phi\cos\omega & \cos\phi\sin\omega \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Lastly, the 7-parameter model will be derived for the points object-space positions.

$\partial F/\partial X$ $\partial F/\partial Y$ $\partial F/\partial Z$

In your HW#4: derive the partial derivatives for the remaining parameters (look into the HW page for more information).

Next Lecture:

- Starting the last Chapter: Operations of Photogrammetry
  - Resection
  - Intersection
  - BBA
  - Single-Ray Backprojection
  - Relative Orientation
  - Absolute Orientation